

Configuration Space Integrals and Universal Vassiliev Invariants over Closed Surfaces

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Abstract

We show the existence of a universal Vassiliev invariant for links in closed surface cylinders by explicit construction using configuration space integrals.

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1 Introduction

The quantization of the algebra of functions on the moduli space of flat connections on a Riemann surface has attracted the attention of a number of authors, including [AGS95], [AS96], [Ati90], [ADPW91], [Buffenoir-Roche], [Fal93] and [Hit90]. The present authors together with Nicolai Reshetikhin developed a new approach in [AMR96b], [Mat97],[AMR96a]. We showed that chord diagrams (see [BN95a], [Vas90]) can be generalized to surfaces and that they form a Poisson algebra of which the algebra of functions on moduli space is a quotient in a natural way. Furthermore, universal Vassiliev invariants ([Kon93],[BNS96],[AMR96a]) allow us to use a simple, geometrically defined multiplication of links to deformation quantize the algebra of chord diagrams. In addition we showed that for punctured surfaces universal Vassiliev invariants exist and that the quantization obtained descends to a quantization of moduli space.

In the present paper we take this program one step further by constructing a universal Vassiliev link invariant for closed surface cylinders. See definition 2.1 and theorem 5.4 for the statement of our main result. The techniques are rather different from those we used before: For punctured surfaces we used the combinatorial construction of universal invariants of tangles due to Bar-Natan [BN95b] and Le and Murakami [Le95] whereas here we generalize the configuration space integral approach of Bott and Taubes [BT94], D.Thurston [Thu95], Altschuler and Freidel [AF96], see also Guadagnini et al. [GMM90] and Bar-Natan [BN95c] as well as Axelrod and Singer [AS94]. Our constructions for punctured and closed surfaces respectively are mutually orthogonal in a literal sense: In our construction for punctured surfaces chords run parallel to the surface whereas in the present construction chords run (almost) vertical. In a subsequent paper we will show that the construction is local on the surface and study the resulting deformation quantization of the algebra of functions on moduli space in detail.

We should point out that we are going to continue to use the notation of [AMR96a] which differs from that preferred by some other authors: We

use ch (rather than \mathcal{A}) to denote the space of chord diagrams and V (rather than Z) to denote a universal Vassiliev link invariant. We also use the terms “Vassiliev invariant” and “finite type invariant” interchangeably.

We wish to thank N.Reshetikhin and D.Thurston and to acknowledge that we profited from a series of lectures by R.Forman at MSRI. Many of our arguments generalize [Thu95] and [AF96] (which in turn are based on [BT94]).

2 Diagrams in 3-manifolds

We refer to [BN95a] for basics concerning chord diagrams and related matters. Let $M = M^3$ be an oriented 3-manifold.

Definition 2.1 *An abstract chord diagram D^a is a graph consisting of disjoint oriented circles $L_I^a, I \in \{1, \dots, n\}$ and disjoint arcs $C_j^a, j \in \{1, \dots, m\}$ such that:*

1. *the endpoints of the arcs are distinct*
2. $\cup_j \partial C_j^a = (\cup_i L_i^a) \cap (\cup_j C_j^a)$

The arcs are called chords, the circles are called the core components of the diagram. Set $L^a = \cup_i L_i^a$ and $C^a = \cup_j C_j^a$.

Definition 2.2 *A chord diagram D in M^3 is a homotopy class of maps from an abstract chord diagram D^a into M . The space generated by chord diagrams modulo 4T-relations is denoted by $ch(M)$.*

More generally we consider trivalent graphs K^a containing a disjoint collection of distinguished oriented circles $\{L_I^a\}$. Set $L^a = \cup_i L_i^a$ and $\Gamma^a = \overline{K^a \setminus L^a}$.

Definition 2.3 *An internal vertex of K^a is a trivalent vertex of Γ^a , an external vertex is a univalent vertex of Γ^a . An abstract Feynman diagram is a trivalent graph K^a together with a cyclic orientation at each internal vertex such that every connected component of Γ^a has at least one external vertex.*

If e denotes the number of edges of Γ^a , u the number of univalent vertices of Γ^a , u_I the number of univalent vertices on L_I^a (so that $u = \sum u_I$) and t the number of trivalent vertices, then $e = \frac{u+3t}{2}$. We define the degree of K^a to be

$$\deg K^a = \frac{u+t}{2} = e - t$$

Note that $t \leq 2 \deg K^a$ and thus $e \leq 3 \deg K^a$. For a chord diagram the degree is just the number of chords. We also use the letter e to denote a general edge of K^a . It should be clear from the context, what the meaning is in a given situation.

We denote the set of abstract Feynman diagrams with u univalent and t trivalent vertices by $\mathcal{F}_{u,t}^a$ and the set of abstract Feynman diagrams of degree n by \mathcal{F}_n^a .

Definition 2.4 *A Feynman diagram is a homotopy class of mappings $K : K^a \rightarrow M^3$ from an abstract Feynman diagram K^a into M , such that $\Gamma = K(\Gamma^a) \subseteq M$ is contractible in M . We call $\Gamma = K(\Gamma^a)$ the graph of the Feynman diagram and $L_K = K(L^a)$ the core of K .*

Definition 2.5 *The vector space generated by Feynman diagrams in M is denoted by $\mathcal{F}(M)$.*

Definition 2.6 *An automorphism of a Feynman diagram K is an automorphism of the abstract Feynman diagram that can be realized by a homotopy of K in M .*

The following generalizes a basic result for \mathbb{R}^3 :

Lemma 2.7 *We have that*

1. $ch(M) \cong \mathcal{F}(M)/STU$
2. $\mathcal{F}(M)/STU$ satisfies AS and IHX.

Proof: The proofs of [BN95a, sect.3.1] apply since Γ is contractible by definition. \square

Remark 2.8 *Note that we have to be careful when applying STU: Two of the diagrams in the STU relation having contractible graph does not imply that the third diagram also has contractible graph.*

3 Configuration spaces

Throughout this section M will be an oriented compact 3-manifold, possibly with boundary. We let M° denote the interior of M .

3.1 Definition and basic properties

Let us briefly recall the basic constructions from [FM94], [BT94], [AS94], [Thu95]:

Let $\mathcal{C}_r(M)$ be the functorial compactified configuration space of r points in M as described by Axelrod and Singer in [AS94]: Let $[r] = \{1, \dots, r\}$. For any subset $S \subset [r]$ (with $|S| > 1$, which we will tacitly assume whenever necessary) we define the diagonal in M^r corresponding to S by

$$\Delta_S = \{(x_1, \dots, x_r) \mid x_i = x_j \text{ for } i, j \in S\} \subseteq M^{\times r}.$$

Now we set

$$\mathcal{C}_r^o(M) = M^r \setminus (\cup_S \Delta_S)$$

and define the space $\mathcal{C}_r(M)$ to be the closure

$$\mathcal{C}_r(M) = \overline{\mathcal{C}_r^o(M)} \subset M^r \times \left(\prod_{S \subset [r]} Bl_S(M) \right)$$

where $Bl_S(M)$ is the (differential-geometric) blow-up of M^r along the diagonal Δ_S , obtained by replacing the diagonal Δ_S with the unit sphere bundle of the normal bundle to Δ_S . Equivalently, $Bl_S(M) = (M^r \setminus \Delta_S) \cup (N(\Delta_S) \setminus \mathbb{R}^+)$ since

$$N(\Delta_S) \setminus \mathbb{R}^+ \cong \left\{ v \in TM^{\oplus S} : |v|^2 = 1, \sum_{I \in S} v_i = 0 \right\} \times M^{\times([r]-S)} \quad (1)$$

By definition of $\mathcal{C}_r(M)$ we have projections from $\mathcal{C}_r(M)$ onto $M^{\times r}$ and $Bl_S(M)$ for $S \subseteq [r]$. Hence, if $x \in \mathcal{C}_r(M)$ is such that its image in $M^{\times r}$ is contained in Δ_S then the projection of x to $Bl_S(M)$ will be contained in $N(\Delta_S) \setminus \mathbb{R}^+$ and we simply write $(x_i)_{I \in S}$ for the element in $TM^{\oplus S}$ which under the above isomorphism corresponds to the projection of x to $Bl_S(M)$.

The space $\mathcal{C}_r(M)$ is a manifold with corners. If M has no boundary, its strata $\mathcal{C}_r^{\mathcal{S}}(M)$ are parametrized by admissible collections of subsets of $[r]$: A collection \mathcal{S} of subsets of $[r]$ is called *admissible* if and $T \in \mathcal{S}$ has at least

two elements and any two sets in \mathcal{S} are either disjoint or one contains the other. The codimension of the stratum is $|\mathcal{S}|$, in particular the codimension one strata are parametrized by subsets T of $[r]$ of cardinality at least two. If M has a nonempty boundary, then there are more strata, since constraining p of the r points to be contained in the boundary has codimension p .

The appropriate relative configuration space for a link L with components L_1, \dots, L_l in M is $\mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))$, introduced in [BT94, Appendix] as the pullback

$$\begin{array}{ccc} \mathcal{C}_{u,t}(L, M) & \longrightarrow & \mathcal{C}_{u+t}(M) \\ \downarrow & & \downarrow \\ \times_i \mathcal{C}_{u_i}(L) & \longrightarrow & \mathcal{C}_u(M) \end{array} \quad (2)$$

where $\mathcal{C}_r(L)$ is defined just as $\mathcal{C}_r(M)$ above and we used the following notation:

$$\mathcal{C}_{u,t}(L, M) = \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))$$

Let $u = \sum_{I=1}^l u_I$ where Γ has u_I univalent vertices incident on L_I . Consider

$$\mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^o = \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l)) \cap \mathcal{C}_{t+u}^0(M)$$

Lemma 3.1 *The space $\mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))$ is the closure in $\mathcal{C}_{t+u}(M)$ of the open configuration space $\mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^o$.*

Proof: The closure has the universal property of the pullback. \square

The stratification is given by

$$\mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^{\mathcal{S}} = \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l)) \cap \mathcal{C}_{t+u}^{\mathcal{S}}$$

for admissible $\mathcal{S} \subseteq \mathcal{P}([u+t])$. In particular the codimension one boundary is given by

$$\partial^1 \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l)) = \bigcup_{T \subset [u+t]} \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^{\{T\}}$$

and

$$\begin{aligned} \partial^1 \left(\mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^{\{T\}} \right) &= \bigcup_{S \subset T} \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^{\{T, S\}} \\ &\quad \bigcup_{S \supset T} \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^{\{T, S\}} \\ &\quad \bigcup_{S \cap T = \emptyset} \mathcal{C}_t(M, (L_1, u_1), \dots, (L_l, u_l))^{\{T, S\}} \end{aligned}$$

Let $\mathcal{C}_{u,t}(M)$ be defined as the space fibering over the space of links in M , with fiber over L equal $\mathcal{C}_{u,t}(L, M)$.

Proposition 3.2 *The space $\mathcal{C}_{u,t}(M)$ is a manifold with corners. Its codimension one faces are the codimension one faces of $\mathcal{C}_{u,t}(M^\circ)$ together with $\partial M \times \mathcal{C}_{u,t-1}(M^\circ)$.*

3.2 Correction bundle

In analogy with [BT94] and [Thu95] we will introduce a certain correction related to anomalous faces. To do this we construct for each $T \subset [u+t]$ a bundle over the unit sphere bundle of TM

$$\pi_T : \mathcal{B}_T \rightarrow S(TM).$$

Let $T_u = T \cap [u]$. For a point x in $\mathcal{C}_{u+t}^{\{T\}}(M)$, we have $(x_i)_{I \in T} \in TM^{\oplus T}$ and we define

$$D(x_i) = \text{span} \{x_I\}$$

Likewise, for $v \in S(TM)$ we define $D(v)$ to be the line $\text{span} \{v\}$ through v .

Definition 3.3 *We set*

$$\mathcal{B}_T^o = \left\{ (v, x) \in S(TM) \times \mathcal{C}_{u+t}^{\{T\}}(M) \mid D(x_i) \subseteq D(v), I \in T_u \right\}$$

and

$$\mathcal{B}_T = \overline{\mathcal{B}_T^o} \subseteq \mathcal{C}_{u+t}^{\{T\}}(M) \subseteq \mathcal{C}_{u+t}(M)$$

For $v \in S(TM)$ we let $B_T(v)$ denote the fiber $\pi_T^{-1}(v)$. The space B_T is constructed such that we get the commutative diagram of subsets

$$\begin{array}{ccc} \mathcal{B}_T & \longrightarrow & \mathcal{C}_{u+t}(M) \\ \uparrow & & \uparrow \\ \mathcal{C}_{u,t}^{\{T\}}(L, M) & \longrightarrow & \mathcal{C}_{u,t}(L, M) \end{array}$$

Proposition 3.4 *The space \mathcal{B}_T is a manifold with corners of dimension*

$$\dim(\mathcal{B}_T) = 3(u+t+1) - 2|T_u|.$$

The projection map

$$\pi_T : \mathcal{B}_T \rightarrow S(TM)$$

is a fibration of manifolds with corners.

Lemma 3.5 *The codimension one boundary of \mathcal{B}_T is given by $\partial^1 \mathcal{B}_T = \cup_S \mathcal{B}_T^S$ where $\mathcal{B}_T^S = \mathcal{B}_T \cap \mathcal{C}_{t+u}^{\{T,S\}(M)}$ and $S \supset T, S \subset T$ or $S \cap T = \emptyset$.*

$$\text{Explicitly, } \mathcal{B}_T^S = \left\{ (v, x) \in S(TM) \times \mathcal{C}_{u+t}^{\{S,T\}} \mid D(x_i) \subseteq D(v), I \in T_u \right\}$$

3.3 Diagrams and strata

Let L be a link in M and let $K^a \in \mathcal{F}_{u,t}^a$ be an abstract Feynman diagram with l core components, where l is the number of components of L . Choose an orientation of each of the edges in Γ^a . Also choose a bijection o between the set of components of L and the set of core components of K^a . Let $\{u_1, \dots, u_l\}$, respectively, be the number of univalent vertices on the core components of K^a so that $\sum_{I=1}^l u_I = u$.

Now choose one of each of the following four orderings:

- o_c An ordering of the components of L .
- o_u An ordering of the u_I univalent vertices on the I 'th core component of K^a compatible with the cyclic order induced from the orientation of the core component.
- o_t An ordering of the t trivalent vertices of K^a .
- o_e An ordering of the e edges of Γ^a .

The orderings o_c and o_u together with the map o specify a component of $\mathcal{C}_{u,t}(L, M)$.

A collection T of vertices of Γ^a defines a subgraph $\Gamma_T^a \subseteq \Gamma^a$ by $\Gamma_T^a = \{e \mid \partial e \subseteq T\}$. Denote by F^d the collection of those $T \subseteq [u+t]$ with Γ_T^a disconnected, by F^a the collection of those $T \subseteq [u+t]$ with $|T| > 1$ and Γ_T^a is a connected component of Γ^a and finally by F^h the collection of those $T \subseteq [u+t]$ with $|T| > 1$ and Γ_T^a connected but not a whole connected component of Γ^a nor is Γ_T^a just an edge of Γ , F^p the collection of those $T \subseteq [u+t]$ for which there is an edge in K^a such that $\partial e = T$.

We have the following decomposition of the codimension one strata of $\mathcal{C}_{u,t}(M)$:

$$\partial \mathcal{C}_{u,t}^1(L, M) = \mathcal{C}_{u,t}^b(L, M) \cup \mathcal{C}_{u,t}^d(L, M) \cup \mathcal{C}_{u,t}^p(L, M) \cup \mathcal{C}_{u,t}^h(L, M) \cup \mathcal{C}_{u,t}^a(L, M),$$

where

$$\begin{aligned}
\mathcal{C}_{u,t}^b(L, M) &= \partial M \times \mathcal{C}_{u,t-1}(L, M^o) \\
\mathcal{C}_{u,t}^p(L, M) &= \cup_{T=Fr} \mathcal{C}_{u,t}^{\{T\}}(L, M) \\
\mathcal{C}_{u,t}^d(L, M) &= \cup_{T \in Fr^d} \mathcal{C}_{u,t}^{\{T\}}(L, M) \\
\mathcal{C}_{u,t}^h(L, M) &= \cup_{T \in Fr^h} \mathcal{C}_{u,t}^{\{T\}}(L, M) \\
\mathcal{C}_{u,t}^a(L, M) &= \cup_{T \in Fr^a} \mathcal{C}_{u,t}^{\{T\}}(L, M)
\end{aligned}$$

are the boundary, disconnected, principal, non-anomalous hidden and anomalous strata, respectively (extending the terminology introduced by Bott and Taubes in [BT94]). By abuse of notation we will also call T boundary, disconnected, principal, hidden or anomalous, respectively.

3.4 Orientations

We define the orientation of $\mathcal{C}_{u,t}(M)$ via an orientation of the abstract Feynman graph, following [AF96]. Let L be a link and let K^a be an abstract Feynman diagram.

Proposition 3.6 *There is an orientation on $\mathcal{C}_{u,t}$ depending on the product of vertex orientations and product of edge orientations of K^a .*

Proof: At a point $x \in \mathcal{C}_{u,t}^o(L, M)$, we have that

$$T_x \mathcal{C}_{u,t}^o(L, M) \cong T_{x_1} L \oplus \dots \oplus T_{x_u} L \oplus T_{x_{u+1}} M \oplus \dots \oplus T_{x_{u+t}} M.$$

Let $X_{x_I} = (X_{x_I}^1, X_{x_I}^2, X_{x_I}^3)$ be an oriented basis for $T_{x_I} M$, $I = u+1, \dots, u+t$, and let $X_{x_I}^1$ be an oriented basis for $T_{x_I} L$, $I = 1, \dots, u$. The orientation element

$$\Omega(K^a) = \bigwedge_{e \in \Gamma^a} \Omega_e,$$

where

$$\Omega_e = X_v^{o_v(e)} \wedge X_u^{o_u(e)} \quad \partial e = (u, v),$$

and o_v is the cyclic order at the internal vertex v , defines an orientation on $T_x \mathcal{C}_{u,t}^o(L, M)$, which orients $\mathcal{C}_{u,t}^o(L, M)$. We then orient all the lower strata of $\mathcal{C}_{u,t}(L, M)$ by inducing their orientation from $\mathcal{C}_{u,t}^o(L, M)$. \square

3.5 Fibre integration

We recall the following fact from [BT94]: If $\pi : B \rightarrow X$ is a fibration with fibre F a compact manifold with corners then for any cycle $c \subseteq X$ the value of the fibre integral $\pi_*(d\omega) = \int_F d\omega$ of an exact form on B is given by

$$\begin{aligned}
 \left(\int_F d\omega \right) (c) &= \int_{\pi^{-1}c} d\omega = \int_{\partial\pi^{-1}c} \omega \\
 &= \int_{\pi^{-1}\partial c} \omega + \int_{\pi^{-1}c \cap \partial B} \omega \\
 &= \int_F \omega (\partial c) + \int_c \left(\int_{\partial F} \omega \right) \\
 &= d \left(\int_F \omega \right) (c) + \left(\int_{\partial F} \omega \right) (c)
 \end{aligned}$$

so that $d \left(\int_F \omega \right) = \int_F d\omega - \int_{\partial F} \omega$. In particular, $d\omega = 0$ and $\int_{\partial F} \omega = 0$ imply $\int_F \omega$ is constant.

4 Configuration space integrals

Let Σ be a closed oriented surface of positive genus. For the remainder of this paper $M = \Sigma \times I$ where $I = [0, 1]$. Let π_Σ denote the projection from M onto Σ and π_I the projection from M onto I .

We choose a hyperbolic metric ρ on Σ if the genus is not 1. In the case where the genus of Σ is 1 we choose a flat metric ρ . For any two points $z_1, z_2 \in \Sigma$ we denote by $\rho(z_1, z_2)$ the length of the shortest geodesic from z_1 to z_2 . For $x, y \in M$, we use the notation $\rho(x, y)$ for $\rho(\pi_\Sigma(x), \pi_\Sigma(y))$. On M we will consider the product metric.

4.1 The two-form on $\mathcal{C}_2(M)$

Fix a natural number n (in the following n will be the degree of a chord diagram) and a small positive real number ε , smaller than the injectivity radius of the metric ρ .

Consider

$$U_n = \{ (x, y) \in M \times M \mid \rho(\pi_\Sigma(x), \pi_\Sigma(y)) < (\varepsilon/3n)^2 \}.$$

Let N be an open neighbourhood of the diagonal in $M \times M$ which satisfies that

$$N \cap U_n = \{(x, y) \in M \times M \mid |\pi_I(x) - \pi_I(y)| < \varepsilon/3n\} \cap U_n.$$

Clearly we can choose N such that $\mathcal{C}_2(M)$ is diffeomorphic to the complement of N in $M \times M$, say by a diffeomorphism $\iota : \mathcal{C}_2(M) \rightarrow M \times M - N$. Note also that $U_n \cap (M \times M - N)$ is disconnected.

Since ε is smaller than the injectivity radius of Σ we have for any $(x, y) \in U_n$ a unique geodesic $\gamma = \gamma_{x,y}$ with $\gamma(0) = \pi_\Sigma(x)$ and $\gamma(1) = \pi_\Sigma(y)$. Using this, we define a map $g : U_n \rightarrow T\Sigma$ by

$$g(x, y) = \gamma'_{x,y}(0) \in T_{\pi_\Sigma(x)}\Sigma.$$

Let Λ_n be a closed 2-form on $T\Sigma$ with support contained in $\{v \in T\Sigma \mid |v| < (\varepsilon/n)^2\}$ which represents the Thom class of $T\Sigma$, see e.g. [BT82, ch.I.6].

Define $\tilde{\omega} = \tilde{\omega}_n$, by

$$\tilde{\omega}_n = \begin{cases} g^*(\Lambda_n) & \text{if } \pi_I(x) > \pi_I(y) \\ -g^*(\Lambda_n) & \text{if } \pi_I(x) < \pi_I(y) \end{cases}$$

Definition 4.1 *Let $\omega = \omega_n$ be the closed 2-form on $\mathcal{C}_2(M)$ given by*

$$\omega = \iota^* \tilde{\omega}_n$$

Remark 4.2 *By construction, ω has support in $\iota^{-1}(U_n)$.*

4.2 The configuration space integral

Let L be a link in M . An abstract Feynman diagram K^a together with the choices specified in section 3.3 defines a map

$$\phi_{K^a} : \mathcal{C}_{u,t}(L, M) \rightarrow \mathcal{C}_2(M)^{\times e}$$

defined on the component of $\mathcal{C}_{u,t}(L, M)$ determined by these choices.

The form ω on $\mathcal{C}_2(M)$ can be pulled back to a two-form on $\mathcal{C}_2(M)^{\times e}$ via any of the projections $\pi : \mathcal{C}_2(M)^{\times e} \rightarrow \mathcal{C}_2(M)$. The wedge product of these pullbacks defines a new form $\omega^e \in \Omega^{2e}(\mathcal{C}_2(M)^{\times e})$.

Consider the form $\phi_{K^a}^* \omega^e$ on the configuration space $\mathcal{C}_{u,t}(L, M)$. For every point $c \in \mathcal{C}_{u,t}(L, M)$ contained in the support of this form there is naturally induced a Feynman diagram K_c^a in M given by mapping the edges of K^a to the unique geodesics between the corresponding pairs of points in M . (The uniqueness follows from the fact that any pair of endpoints of an edge is contained in U_n .)

Let S_l denote the set of possible choices for the map o (so S_l is in bijective correspondence with the symmetric group on l elements).

Definition 4.3 Let $\omega_{\mathcal{C}_{u,t}}^{K^a} \in \Omega^{2e}(\mathcal{C}_{u,t}(L, M), \mathcal{F}(M))$ be given by

$$\omega_{\mathcal{C}_{u,t}}^{K^a}(c) := \sum_{o \in S_l} \frac{\phi_{K^a}^* \omega^e(c)}{|Aut K_c^a|} K_c^a$$

for $c \in \mathcal{C}_{u,t}(L, M)$.

The reason why we only sum over the choices of o will be explained below. Clearly, the class of K_c^a is locally constant and therefore constant on the connected components of $\text{supp}(\phi_{K^a}^* \omega^e) \subseteq \mathcal{C}_{u,t}(L, M)$. Thus $\omega_{\mathcal{C}_{u,t}}^{K^a}$ is integrable, and by compactness we see that $\int_{\mathcal{C}_{u,t}} \omega_{\mathcal{C}_{u,t}(L, M)}^{K^a}$ is a finite sum of Feynman diagrams in M , hence it specifies an element in $\mathcal{F}(M)$.

Since ω is a two form, this integral is independent of the chosen total ordering o_e of the edges of Γ^a . From $\omega_{(y,x)} = -\omega_{(x,y)}$ it follows that $\omega_{\mathcal{C}_{u,t}}^{K^a}$ depends on the product of the orientations of the edges. Thus proposition 3.6 implies that $\int_{\mathcal{C}_{u,t}} \omega_{\mathcal{C}_{u,t}}^{K^a}$ is independent of the choice of edge orientations. Changing the ordering o_t of the internal vertices of K^a or the ordering of the external vertices o_u on any of the components, results in an orientation preserving self-diffeomorphism of $\mathcal{C}_{u,t}(L, M)$ taking the form for one orientation to the form for the other, hence the integral is independent of these orderings.

Hence we can now make the following definition:

Definition 4.4 We define for any abstract trivalent Feynman diagram K^a

$$\tilde{V}(L, K^a) = \int_{\mathcal{C}_{u,t}(L, M)} \omega_{\mathcal{C}_{u,t}}^{K^a}$$

We will consider the variation of this integral with respect to the link L . Using arguments similar to those of [BT94], [Thu95] and [AF96] we calculate $\delta \int_{\mathcal{C}_{u,t}(L, M)} \omega_{\mathcal{C}_{u,t}(L)}^{K^a}$ using the properties of the form $\omega_{\deg K}$ chosen above and the definition of the automorphism group of a diagram.

4.3 Variation of the function \tilde{V}

Let us define

$$\tilde{V}_n(L) = \sum_{K^a \in \mathcal{F}_n^a} \tilde{V}(L, K^a)$$

and

$$\tilde{V}(L) = \sum_{n=0}^{\infty} \frac{\tilde{V}_n(L)}{2^n}$$

thought of as a function on the space of links in M with values in $\overline{ch}(M)$. We want to compute the derivative of the function \tilde{V} .

By the version of Stokes theorem presented in section 3.5,

$$\delta \tilde{V} = \delta^b \tilde{V} + \delta^d \tilde{V} + \delta^p \tilde{V} + \delta^h \tilde{V} + \delta^a \tilde{V}$$

where

$$\begin{aligned} \delta^b \tilde{V} &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{K^a \in \mathcal{F}_n^a} \int_{\mathcal{C}_{u,t}^b(L)} \omega_{\mathcal{C}_{u,t}(L)}^{K^a} \\ \delta^d \tilde{V} &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{K^a \in \mathcal{F}_n^a} \int_{\mathcal{C}_{u,t}^d(L)} \omega_{\mathcal{C}_{u,t}(L)}^{K^a} \\ \delta^p \tilde{V} &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{K^a \in \mathcal{F}_n^a} \int_{\mathcal{C}_{u,t}^p(L)} \omega_{\mathcal{C}_{u,t}(L)}^{K^a} \\ \delta^h \tilde{V} &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{K^a \in \mathcal{F}_n^a} \int_{\mathcal{C}_{u,t}^h(L)} \omega_{\mathcal{C}_{u,t}(L)}^{K^a} \\ \delta^a \tilde{V} &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{K^a \in \mathcal{F}_n^a} \int_{\mathcal{C}_{u,t}^a(L)} \omega_{\mathcal{C}_{u,t}(L)}^{K^a}. \end{aligned}$$

Theorem 4.5 *The variation of \tilde{V} equals the variation over the anomalous faces:*

$$\delta \tilde{V} = \delta^a \tilde{V}$$

Proof: By the above, we need to establish that $\delta^b \tilde{V} = 0$, $\delta^d \tilde{V} = 0$, $\delta^p \tilde{V} = 0$ and $\delta^h \tilde{V} = 0$.

The variation over the boundary faces $\delta^b \tilde{V}$ is zero. This follows since $\omega_{(x,y)}$ depends on $\pi_\Sigma(x), \pi_\Sigma(y)$ only up to sign. Hence

$$\int_{x \in \Sigma \times \{1\}} \int_{\mathcal{C}_{u,t-1}(\Sigma \times (0,1))} \phi_{K^a}^* \omega^e = - \int_{x \in \Sigma \times \{0\}} \int_{\mathcal{C}_{u,t-1}(\Sigma \times (0,1))} \phi_{K^a}^* \omega^e$$

which shows that these contributions cancel.

If T is disconnected ϕ_{K^a} factors through a codimension three subspace so that $\phi_{K^a}^* \omega^e = 0$.

The variation of \tilde{V} coming from the hidden faces vanishes: By the symmetry arguments in [Thu95] there are either orientation reversing automorphisms of the strata preserving ω^K or orientation preserving automorphisms of the strata mapping ω^K to $-\omega^K$ (this uses $\omega_{(y,x)} = -\omega_{(x,y)}$).

Finally, the following section will establish that $\delta^p \tilde{V} = 0$. \square

The variation on anomalous faces $\delta^a \tilde{V}$ might be nonzero. Let $T \subset [u+t]$ define an anomalous face. Let $I : L \rightarrow S(TM)$ be the natural tangential inclusion map. Consider the diagram

$$\begin{array}{ccccc} (\mathcal{C}_2(M))^e & \xlongequal{\quad} & (\mathcal{C}_2(M))^e & \xlongequal{\quad} & (\mathcal{C}_2(M))^e \\ \uparrow \phi_{K^a} & & \uparrow \phi_{K^a} & & \uparrow \psi_{K^a} \\ \mathcal{C}_{u,t} & \longleftarrow & \mathcal{C}_{u,t}^{\{T\}} & \longrightarrow & B_T \\ \downarrow & & \downarrow & & \downarrow \pi_T \\ L^{\times u} \times M^{\times t} & \longleftarrow & L & \xrightarrow{I} & S(TM) \end{array} \quad (3)$$

We obtain a form $\omega_{B_T}^{K^a}$ on B_T by replacing ϕ_{K^a} by ψ_{K^a} in definition 4.3. If we let $\Omega_T^{K^a}$ be the 2-form on $S(TM)$ obtained by pushing $\omega_{B_T}^{K^a}$ forward to $S(TM)$, i.e.

$$\Omega_T^{K^a}(v) = \int_{B_T(v)} \omega_{B_T}^{K^a},$$

then clearly

$$\delta^a \int_{\mathcal{C}_{u,t}} \omega_{\mathcal{C}_{u,t}}^{K^a} = \int_{\mathcal{C}_{u,t}^{\{T\}}} \omega_{B_T}^{K^a}|_{\mathcal{C}_{u,t}^{\{T\}}} = \int_L i^* \Omega_T^{K^a} \quad (4)$$

In section 5.2 we will adjust \tilde{V} to obtain a function which also has zero variation over the anomalous faces, hence is an invariant of links.

4.4 Variation on principal faces

Recall that we are using e both as a variable running through the edges of K^a and as the number of edges in Γ^a .

Collapsing an edge e of an abstract Feynman diagram K^a leads to a graph $\delta_e K^a$ with one 4-valent vertex moreover, e specifies a principal face $\mathcal{C}_{u,t}^{\partial e}(L, M)$. It is easy to extend the discussion in section 4.2 to this case (for details see [AF96, prop.3]). In particular, one $\delta_e K^a$ induces an orientation on $\mathcal{C}_{u,t}^{\partial e}(L, M)$ and one gets a map $\phi_{\delta_e K^a} : \mathcal{C}_{u,t}^{\partial e}(L, M) \rightarrow C_2(M)^{\times e}$ such that

$$\omega_{\mathcal{C}_{u,t}^{\partial e}}^{K^a}|_{\mathcal{C}_{u,t}^{\partial e}(L, M)}(c) = \sum_{o \in S_l} \frac{\phi_{\delta_e K^a}^* \omega^e(c)}{|\text{Aut } K_c^a|} K_c^a$$

Proposition 4.6 *The orientation defined in proposition 3.6 and the orientation induced from $\delta_e K^a$ on $\mathcal{C}_{u,t}^{\partial e}(L, M)$ agree.*

Proof: As in [AF96, prop.3] \square

Lemma 4.7 *Let Γ_x be a graph with one four-valent vertex x , all other vertices trivalent. Then there are at most three trivalent graphs $\Gamma_x^I, I \in \{1, 2, 3\}$, such that $\delta_{e_I} \Gamma_x^I = \Gamma_x$ for edges $e_I \in \Gamma_x^I$.*

Proof: Holds as in [AF96, lemma 1] since this is a local statement. \square

Lemma 4.8 *K a trivalent graph with edges e, f . Then $\delta_e K \cong \delta_f K$ iff there is an $\sigma \in \text{Aut}(K)$ such that $\delta_e(\sigma(K)) = \delta_e(K), \sigma(e) = f$*

Proof: Obvious. (as [AF96, lemma 2]) \square

Proposition 4.9 *$\delta_e K \cong \delta_f K$ iff $\sigma(K) = \pm(\delta_e K)^I, \sigma(f) = e$ for some $I, \sigma \in \text{Aut}(K)$*

Proof: As [AF96, prop.1]. \square

Lemma 4.10 *If e is admissible and collapsing e does not result in two vertices being connected by two different edges, then $P_I^\pm(K, e) \cap P_j^\pm(K, e) = \emptyset$*

Proof: [AF96, lemma 5] is stronger. \square

Lemma 4.11 $\# \{f | \delta_f K \cong \pm \delta_e K\} = \# \{I\} \times \frac{|Aut_+ K|}{|Aut_+ \delta_e K|}$

Proof: Follows from the above lemmata, as in [AF96, lemma 4]. \square

From the above it now follows that

Lemma 4.12 *The variation on principal faces is given by*

$$\delta^p \tilde{V} = \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{\substack{K^a \in \mathcal{F}_n \\ e \in K^a}} \frac{1}{2^n} \int_{\mathcal{C}_{u,t}^{\partial e}(L,M)} \omega_{\mathcal{C}_{u,t}}^{\delta_e K^a}$$

Lemma 4.13 *If there is an orientation reversing automorphism of K or $\delta_e K$ then $[K] = 0$ or $\int_{\mathcal{C}_{u,t}^{\partial e}(L,M)} \omega_{\mathcal{C}_{u,t}}^{\delta_e K^a} = 0$.*

Proof: Obvious. \square

Lemma 4.14 *One gets that $\int_{\mathcal{C}_{u,t}^{\partial e}(L,M)} \omega_{\mathcal{C}_{u,t}}^{\delta_e K^a} = 0$ if collapsing e does result in two vertices being connected by two different edges.*

Proof: ϕ_{K^a} factors through $(\mathcal{C}_2(M))^{e-1}$, hence $\phi_{K^a}^* \omega^e = 0$. \square

Theorem 4.15 *The variation on principal faces is zero: $\delta^p \tilde{V} = 0$.*

Proof: Follows from the last three lemmata as in [AF96, theorem 1] since all the arguments are local. \square

5 The invariant

5.1 Correction terms

Here we will construct the correction factor to cancel the variation over the anomalous faces. From equation (4) we see that we need a factor whose variation is $\sum_{K^a \in \mathcal{F}_n} \sum_{T \in F^a(K^a)} \int_L i^* \Omega_T^{K^a}$. To construct such a correction we want to show that the 2-form $\Omega_n = \sum_{K^a \in \mathcal{F}_n} \sum_{T \in F^a(K^a)} \Omega_T^{K^a}$ is exact. We will do this only on an open dense submanifold $S'(TM)$ of $S(TM)$ containing a neighbourhood of L , but this suffices for our purposes.

Our first step is

Proposition 5.1 *The 2-form Ω_n is closed.*

Proof: By lemma 3.5,

$$\delta\Omega_n = \sum_{K^a \in \mathcal{F}_n^a} \sum_{T \in F^a(K^a)} \sum_S \int_{B_T^S(v)} \omega_{B_T}^{K^a}$$

where one of the following holds:

1. S defines a boundary face
2. $S \subset T$
3. $S \supset T$
4. $S \cap T = \emptyset$

If S is boundary the contribution is zero as above.

If $S \subset T$ then S is principal or hidden and the contribution is zero.

If $S \supset T$ then S is disconnected and does not contribute.

Hence we see that the contributions to $\delta\Omega_n$ all vanishes unless $S \cap T = \emptyset$ and S is anomalous.

Now we note that S contains at least one univalent vertex v_0 (since S is a connected component of Γ) that in $\mathcal{C}_{u,t}^{\{T,S\}}(L, M)$ runs over the whole tangent space of $T_x M$ if the points parametrized by S collapse at x . Let v_1 be the vertex which is connected to v_0 by a edge in Γ . Reflection of v_0 in v_1 is an automorphism θ of B_T^S such that $\theta^* \omega = -\omega$ and $\theta^* \int_{B_T^S(v)} \omega = - \int_{B_T^S(v)} \omega$ implies $\int_{B_T^S(v)} \omega = 0$. This completes the proof that $\delta\Omega_n = 0$. \square

Now we proceed as follows:

Let \mathbf{n} be a section of $S(TM)$. We shall now restrict our self to links $L \subset M$ such that $\mathbf{n}(M) \cap TL = \emptyset$ inside $S(TM)$. In other words \mathbf{n} induces a framing of L .

Choose a point $x \in \Sigma$ not on the projection of the link L and let

$$S'(TM) = S(TM) - \left\{ \mathbf{n}(M) \cup S(TM) \big|_{\pi_\Sigma^{-1}(x)} \right\}.$$

By construction $S'(TM)$ is homotopy equivalent to $\Sigma - \{x\}$, hence $H^2(S'(TM)) = 0$. Hence, if we let $\Omega'_n = \Omega_n|_{S'(TM)}$, then Ω'_n is exact. Now pick cycles b_I representing a basis for $H_1 \Sigma$.

Definition 5.2 *Let α_n be such that*

$$\begin{aligned}\Omega'_n &= d\alpha_n \\ \alpha_n(b_I) &= 0.\end{aligned}$$

Any two choices for α_n will be cohomologous: $d(\alpha_n - \alpha'_n) = \Omega_n - \Omega'_n = 0$ and $(\alpha_n - \alpha'_n)(b_I) = 0 \forall I$ so that $[\alpha_n - \alpha'_n] = 0 \in H^1(S'(TM))$.

By the version of Stokes theorem in section 3.5 we see that

$$\delta \int_L \alpha_n = \int_L d\alpha_n = \int_L \Omega_n \quad (5)$$

and $\int_L (\alpha_n - \alpha'_n) = \int_L df_n = \int_{\partial L} f_n = 0$ shows that it is independent of α_n . Since \mathbf{n} is only allowed to vary continuously in the complement of $I(L)$ and since $H^2(\Sigma \setminus \{x\}) = 0$, it is clear that the cohomology class of α_n is independent of the choice of framing \mathbf{n} .

Furthermore $\int_L \alpha_n$ is independent of the choice of x : Consider two choices of points x, y and the corresponding forms α_n^x, α_n^y . The first homology of $S(TM) \setminus \left\{ \mathbf{n}(M) \cup S(TM)|_{\pi_\Sigma^{-1}(\{x,y\})} \right\}$ is generated by $\{b_I\}$ and a small loop c_y around y on Σ . By assumption $\forall \alpha_n^x(b_I) = \alpha_n^y(b_I)$ for all I , in addition $\alpha_n^x(c_y) = 0$ since c_y is homologous to 0 in the domain of α_n^x . Since also $\alpha_n^y(c_y) = \alpha_n^y(c_x) = 0$ (because c_y is homologous to $c_x \bmod \{b_I\}$) we see that α_n^x is cohomologous to α_n^y near L and thus $\int_L \alpha_n^x = \int_L \alpha_n^y$.

5.2 Definition of the invariant

Let $\overline{ch(M)}$ denote the completion of $ch(M)$ with respect to the filtration defined by degree.

Definition 5.3 *Let $V_n(L)$ be defined by*

$$V_n(L) = \tilde{V}_n(L) - \int_L \alpha_n.$$

Theorem 5.4 *The sum*

$$V(L) = \sum_n \frac{1}{2^n} V_n(L)$$

defines an element of $\overline{ch(M)}$. The map $L \mapsto V(L) \in \overline{ch(M)}$ is a universal finite type invariant for links in M .

Proof:

By the definition of U_n we know that only those imbeddings of K contribute where for any connected component \bar{K} of K the projection $\pi_\Sigma(\bar{K})$ is contained in some disk of radius at most $e\left(\frac{\varepsilon}{3\deg K}\right)^2 \leq \frac{\varepsilon^2}{3\deg K}$ on Σ , in particular the graph of Γ is contractible. From this, the finiteness of the sum defining V and from $\mathcal{F}/STU \cong ch$ (lemma 1) it follows that the sum is well-defined as a sum of chord diagrams.

We have that

$$\delta V_n = \delta \tilde{V}_n - \delta \int_L \alpha_n = \delta^a \tilde{V}_n - \int_L \Omega_n = 0$$

by the construction of Ω_n in the previous section.

We are left with showing that $V(L)$ is in fact a universal finite type invariant. This will be done in the next section. \square

5.3 Finite type and universality

Consider a resolution of a singular knot $\sum_{\eta \in \{\pm 1\}^n} (-1)^\eta L_\eta$ of degree n . Let the L_η be equal outside balls $B_I, I = 1, \dots, n$, of size ε around the singularities. By invariance of V under isotopy we can also assume they are almost planar (contained in $\Sigma \times \{\frac{1}{2}\}$), all the $\pi_\Sigma(L_\eta)$ are equal and the distance of the balls satisfies $|B_I, B_j|_\mu > 3\varepsilon/n$. If for some I no edge of Γ ends in B_I then clearly $\sum (-1)^\eta \omega_\eta^{K^a} = 0$.

Now we consider the correction term: Assume L^1 and L^2 are singular links of degree $n-1$ that differ by a crossing switch away from the B_I and let $\chi(t)$ be a homotopy from L^1 to L^2 , then

$$\begin{aligned} \sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_{L_\eta^1} \alpha_{\deg K^a} - \sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_{L_\eta^2} \alpha_{\deg K^a} \\ = \sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_t \int \chi(t)^* \Omega'_{\deg K^a}. \end{aligned}$$

If $2\deg K^a < n-1$ then for each imbedding of K^a either the projection of some edge has length greater than $(\varepsilon/3n)^2$ or at least one $B_I, I = 1, \dots, n-1$ does not contain a univalent vertex. In either case

$$\sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_{\text{Im}\chi(t)} \Omega'_{\deg K^a} = 0$$

for each t , so that

$$\begin{aligned} \sum_{\eta' \in \{\pm 1\}^n} (-1)^{\eta'} \int_{L_{\eta'}} \alpha_{\deg K^a} = \\ \sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_{L_\eta^1} \alpha_{\deg K^a} - \sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_{L_\eta^2} \alpha_{\deg K^a} = 0. \end{aligned}$$

This shows that $V(L)$ of finite type.

Finally, we show that $V(L)$ is universal: We observe that any chord that contributes to $\int \omega^{K^a}$ projects into a small disk in Σ (by remark 4.2) so that the integral decomposes as a product of integrals over disjoint domains for each double point of the projection since $|B_I, B_j|_\mu > 3\varepsilon/n \geq \frac{e\varepsilon}{n \deg K}$ (cf. [AF96, proof of theorem 5]), hence if a component of Γ intersects two different B_I then the contribution to the invariant vanishes. Let γ be the number of connected components of Γ^a . Since $\gamma \leq \deg K^a$ and $\gamma = \deg K^a$ iff K^a has no trivalent vertex it follows that $\sum_{\eta \in \{\pm 1\}^n} (-1)^\eta \tilde{V}_I(L_\eta) = 0$ for $i < n$. Furthermore, α can contribute only if K^a has at least one trivalent vertex and $\sum_{\eta \in \{\pm 1\}^{n-1}} (-1)^\eta \int_{\text{Im}\chi(t)} \Omega'_I = 0$ for $I = 1, \dots, n$ by the argument above so that $\sum_{\eta \in \{\pm 1\}^n} (-1)^\eta \int_{L_\eta} \alpha_I = 0$ for $i < n$.

If $e = \deg K^a = n$ (i.e. no trivalent vertex) then $\tilde{V}(L_\eta, K^a) \neq 0$ only if there is exactly one chord in each B_I , which necessarily runs almost vertical. The coefficient will be 1 by the defining properties of the Thom class Λ_n entering in the construction of ω_n . To see that the correction term will be zero for $\deg K^a = n$ we note that we can assume that $\chi(t)$ moves only in the vertical direction. By the definition of ω again, we see that Ω' has no component in the vertical direction, thus we effectively integrate a two-form over a manifold of dimension one. Hence the anomalous contribution is zero for $\deg K^a = n$.

This concludes the proof of the main theorem 5.4.

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